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EE 364 - Spring 2026

# Week 5
# Supplemental Notes

Kolmogorov - Foundations  Grundbegiffe (1918)

$\Omega \longleftrightarrow$ measure space

$(\Omega, \mathcal{Q})$ sigma-algebra : abstraction of sample space and assignment of probability to events.

$P \longleftrightarrow$ CAT measure

* r.v. $X \longleftrightarrow$ Measurable function (PAM)

" random experiment "

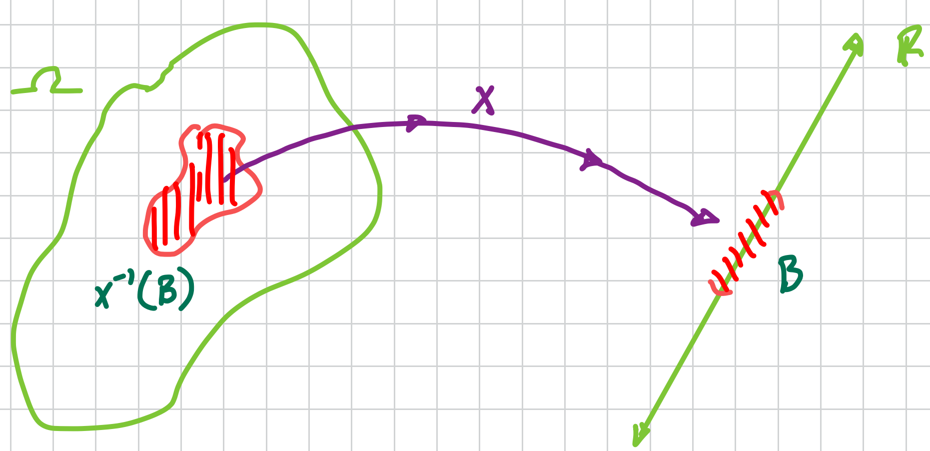
**Defn:** $X : \Omega \longrightarrow \mathbb{R}$ is a random variable (r.v.)

on $(\Omega, \mathcal{Q})$

iff PAM: Pullbacks are Always Measurable

iff $\forall B \in \mathcal{B}(\mathbb{R})$ : $\boxed{X^{-1}(B) \in \mathcal{Q}.}$

intervals in $\mathbb{R}$.



$\therefore$ $P(X^{-1}(B))$ defined if $X^{-1}(B) \in \mathcal{A}$ since

since $P: \mathcal{A} \longrightarrow [0,1]$

$\therefore$ $X$ is <u>not</u> a random variable if $\exists B \in \mathcal{B}(\mathbb{R})$ w/ $X^{-1}(B) \notin \mathcal{A}$

Ex: flip coin twice $\quad \Omega = \{H,T\} \times \{H,T\}$ $\quad P(H) = p$

$(\mathcal{A} = 2^{\Omega})$ $\quad P(T) = 1-p = q$

Define: $X$ = # heads in 2 flips

$Y = (\# H)^2$

$Z = (\# H)^3 - 1$

uncountable infinite possible r.v.

Outcomes

| | TT | TH | HT | HH |
|---|---|---|---|---|
| X | 0 | 1 | 1 | 2 |
| Y | 0 | 1 | 1 | 4 |
| Z | -1 | 0 | 0 | 7 |

notation doesn't make sense. "function = constant"?

$\therefore$ $P[X=0] = P[X^{-1}(\{0\})] = P[\{TT\}] = q^2$

$P[X=1] = P[X^{-1}(\{1\})] = P[\{TH, HT\}] = 2pq$

$P[X=2] = P[X^{-1}(\{2\})] = P[\{HH\}] = p^2$

$P[X=3] = P[X^{-1}(\{3\})] = P[\emptyset] = 0$

Also: $P[Z=0] = P[Z^{-1}(\{0\})] = P[\{TH, HT\}] = 2pq.$

**Ex:** (Indicator function) $I_A : \Omega \to \{0,1\}$.

with $A \subset \Omega$ and $A \in \mathcal{Q}$. Define

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A. \\ 0 & \text{if } \omega \notin A. \end{cases}$$

$\therefore$ $I_A^{-1}(\{0\}) = A^c \in \mathcal{Q}$ (by cut)

$I_A^{-1}(\{1\}) = A \in \mathcal{Q}$

$\therefore$ $I_A$ is a r.v. (is "measureable")

Note: "Fuzzy" set $\longleftrightarrow$ $I_A : \Omega \to \underset{\sim}{[0,1]}$ vs. $\{0,1\}$.

**Ex:** PAM can fail. Define $\Omega = [0,10]$

$\mathcal{Q} = \{\phi, [0,3), [3,10], \Omega\}$.

$X : \Omega \to \Omega$

$X(\omega) = \omega$ if $\omega \in \Omega$ (identity function)

Range S.A. $\mathcal{B}([0,10])$

pick $B = (2,4) \in \mathcal{B}([0,10])$

$\therefore$ $X^{-1}(B) = X^{-1}((2,4)) = (2,4) \notin \mathcal{Q}$.

$\therefore$ $X$ is NOT a r.v. (PAM fails)

**Defn:** The probability <u>mass</u> function (p.m.f.)

$p_X(x_k) = \underline{P[X = x_k]}$ where $x_k$ are the outcomes from random experiment with <u>countable sample space</u>

Two properties: (1) $p_X(x_k) \geq 0 \quad \forall x_k \in \Omega$

(2) $\sum_{x_k \in \Omega} p_X(x_k) = 1.$

**Ex:** binomial "distribution" (random variable)

$$P[\underbrace{X = k}_{\text{k success in n flips}}] = \binom{n}{k} p^k (1-p)^{n-k}$$

**Defn:** <u>Cumulative Distribution Function</u> (CDF)

Say $X : \Omega \to \mathbb{R}$ is a r.v. on p.s. $(\Omega, \mathcal{A}, P)$.

Then CDF $F_X(x) \triangleq P(X \leq x)$

$$= P(\{w \in \Omega : X(w) \leq x\})$$

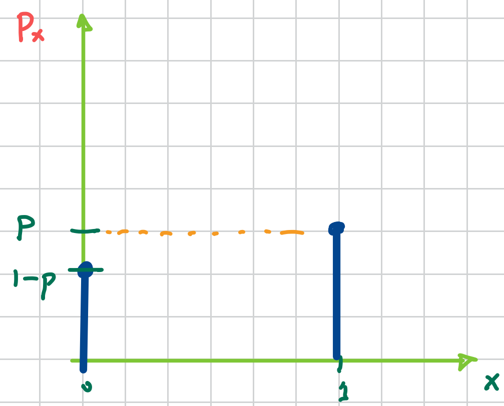
$$= P(X^{-1}((-\infty, x]))$$

Note: CDF $F_X$ always exists.

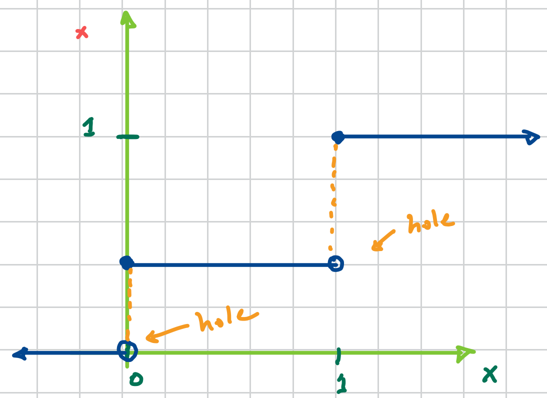
| <u>Random Variable</u> | <u>Realization</u> |
|---|---|
| • X | x |
| • function | constant |
| • foot | footprint. |

**Note:** Cumulative distribution function (CDF) for discrete random variable $X$: $F_X(x) = P[X \le x] = \sum_{x_k \in X} p_X(x_k)$

**Ex:** Bernoulli random variable $P[X=1] = p$, $P[X=0] = 1-p$.




pmf, $p_X(x)$




CDF, $F_X(x)$

**Note:** pmf is a discrete function, defined only for discrete argument $x_k \in \Omega$. CDF is defined $\forall x \in \mathbb{R}$.

## Properties of the CDF:

1. $F_X(-\infty) = 0$
2. $F_X(u_1) \le F_X(u_2)$ if $u_1 \le u_2$ — $F_X$ is increasing
3. $F_X(\infty) = 1$
4. $F_X(b) = \lim_{h \to 0^+} F_X(b+h)$ — $F_X$ is right continuous

4b. $P[X=b] = \underbrace{\lim_{h \to 0^+} F_X(b+h)}_{\text{approach from right}} - \underbrace{\lim_{h \to 0^-} F_X(b+h)}_{\text{approach from left}}$ } "jump"

# Working with the CDF:

disjoint

Consider event $\{X \le u_1\} = \{X \le u_0\} \cup \{u_0 < X \le u_1\}$ for $u_0 < u_1$

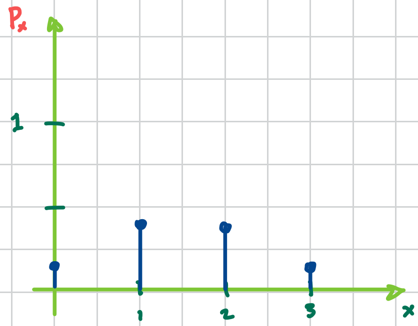
$$\therefore \; P[\underbrace{\{X \le u_1\}}_{F_x(u_1)}] = P[\underbrace{\{X \le u_0\}}_{F_x(u_0)}] + P[\{u_0 < X \le u_1\}].$$

$$\therefore \; \boxed{P[u_0 < X \le u_1] = F_x(u_1) - F_x(u_0)}$$

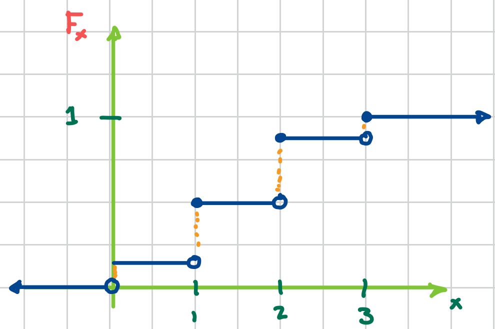
$$\therefore \; P[u_0 \le X \le u_1] = P[X = u_0] + P[u_0 < X \le u_1]$$

$$= F_x(u_1) - F_x(u_0) + p_x(u_0)$$

Ex: Count # heads in 3 fair coin flips $\qquad X \sim b(3, 0.5)$




pmf, $p_x(x)$




CDF, $F_x(x)$

Measuring the size of sets

<u>Defn</u>: $f: X \to Y$ is 1-to-1 ("Injective") iff

$$\forall x \, \forall z : f(x) = f(z) \longrightarrow x = z.$$

<u>Defn</u>: $f: X \to Y$ is ONTO ("Surjective") iff

$$\forall y \in Y \; \exists x \in X : y = f(x) \qquad (f(X) = Y)$$

<u>Defn</u>: $f: X \to Y$ is a BIJECTION ("1-to-1 correspondence")

iff $f$ is 1-to-1 and onto.

$|A|$ = cardinality ("size") of set $A$.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ integers

$\mathbb{Z}^+ = \mathbb{N}$

$\mathbb{R} = (-\infty, \infty)$ reals

<u>Defn</u>: $A$ is finite iff $A \underset{\text{onto}}{\overset{1\text{-to-}1}{\longleftrightarrow}} S$ for some $S \subset \mathbb{N}$

such that $A = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ (or $A = \emptyset$)

Else $A$ is infinite.

<u>Fact</u>: $A$ infinite $\longleftrightarrow$ $A \underset{\text{onto}}{\overset{1\text{-to-}1}{\longleftrightarrow}} B$ for some $B \subset A$ and $B \neq A$.

$\therefore$ infinite $\longleftrightarrow$ $\sim$finite

**Cantor's theorem:** $|X| < |2^X|$

**Defn:** $A$ is denumerable iff $A \xleftrightarrow[\text{onto}]{\text{1-to-1}} \mathbb{N}$ (e.g. $\mathbb{Z}$)

**Defn:** $A$ is countable iff (1) $A$ is finite, or

(2) $A$ is denumerable

**Facts:** $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$ (aleph-nought)

$|\mathbb{R}| = |2^{\mathbb{Z}}| = c = \aleph_1$

power of the continuum

$|A| = \aleph_k \Longrightarrow |2^A| = \aleph_{k+1}$ $(k = 0, 1, \ldots)$

**Cantor's Continuum Hypothesis** There is no $\omega$ such that

$\aleph_k < \omega < \aleph_{k+1}$

**Ex:** $(0,1)$ same size as $\mathbb{R}^+$

$f(x) = \frac{1}{1 + e^{-x}}$ (note: adding 2-points $0, 1 \longrightarrow [0,1]$ no 1-to-1 onto map w/ $\mathbb{R}^+$)

Aside: $\pm \infty$ not real numbers

# Continuous probability distributions ("densities")

Suppose sample space $\Omega$ not countable   e.g., $\Omega = [0,1]$

$\hookrightarrow$ $\therefore$ CDF continuous.   $\Omega = \mathbb{R}$

**Defn:** $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$

iff $\forall \varepsilon > 0 \; \exists \delta > 0 : |x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon.$

inputs close ($x, x_0$) $\longrightarrow$ outputs close ($f(x), f(x_0)$)

- $f$ is continuous at $a$ ($\because$ left + right continuous)

reminder: CDF only requires right continuous

$\longleftrightarrow \lim_{x \to a} f(x) = f(a)$   $\uparrow$ if $f$ defined at $a$

$\longleftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : |x - a| < \delta \to |f(x) - f(a)| < \varepsilon.$

**Defn:** $f$ is continuous iff it is continuous at $\forall x_0 \in \mathbb{R}$.

**Fact:** $P[X = b] = \underbrace{\lim_{h \to 0^+} F_X(b+h)}_{\text{right limit}} - \underbrace{\lim_{h \to 0^-} F_X(b+h)}_{\text{left limit}}$

$= 0$ if $F_X$ continuous

$\uparrow$ not "close to zero" or "very small", identically 0

$\therefore$ If $X$ is a continuous random variable ($\because$ CDF continuous)

then $P[X = x] = 0 \quad \forall x \in \mathbb{R}$.

i.e. $P[X = a] = P[X = b] = 0$

$$\therefore \boxed{P[a < X \le b] = P[a \le X \le b] = P[a \le X < b] = P[a < X < b]}$$

Q: If $P[X=x]=0$ for all $x$ how to describe probability of an outcome? (e.g., $P[\underbrace{\text{height}=6\text{ ft}}_{=0}]$)

A: Use density of probability "near" the outcome (e.g., $P[\text{height}\approx 6\text{ ft}]$)

*

**Thm:** CDF $F$ is absolutely continuous iff $\exists$ probability density function (pdf) $f$:

$$F(x)=\int_{-\infty}^{x} f(t)\,dt. \quad \therefore \text{①} \boxed{f=F'=\frac{dF}{dx}}$$

$$\text{②} \int_{-\infty}^{\infty} f(x)\,dx=1$$

**Defn:** The probability density function (pdf) of a continuous random variable $X$: $\boxed{f_X(x)=\frac{dF_X}{dx}}$ (if it exists) iff $X$ is "absolutely continuous"

$$\therefore F_X(x)=\int_{-\infty}^{x} f_X(t)\,dt.$$

Fundamental theorem of calculus.

$$\int_a^b f(x)\,dx = \underbrace{F(b)-F(a)}_{\text{"anti-derivative"}}$$

i.e. $P[a\le X\le b]=F_X(b)-F_X(a)=\underbrace{\int_a^b f_X(x)\,dx}_{\text{if it exists.}}$

- behavior of interior is completely determined by behavior at boundary.
- simplifies operation of probability because it hides underlying complexity.

# Differentiable functions

**Defn:** function $f$ is differentiable at $a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a) \quad \text{exists.}$$

**Ex:** $f(x) = |x|$ continuous everywhere

not differentiable (at $x = 0$)

**Thrm:** Differentiable $\longrightarrow$ Continuous

**Pf:** $f(x) = f(x) + f(a) - f(a) = \left( \frac{f(x) - f(a)}{x - a} \right)(x - a) + f(a)$

$$\therefore \lim_{x \to a} f(x) = \lim_{x \to a} \left[ \left( \frac{f(x) - f(a)}{x - a} \right)(x - a) + f(a) \right]$$

$$= \underbrace{\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right)}_{= f'(a) \text{ exists by assumption}} \cdot \underbrace{\lim_{x \to a} (x - a)}_{= 0} + f(a)$$

$$= f(a)$$

$$\therefore \lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f$ continuous at $a$.

$\therefore$ $f$ continuous (since $\forall a$)

QED.

**Insight:** The pdf specifies the density of probability.

Ex: $P[x < X \leq x + \Delta x] = F_X(x+\Delta x) - F_X(x)$

$$= \Delta x \, \frac{F_X(x+\Delta x) - F_X(x)}{\Delta x}$$

$\therefore$ for small h. $\simeq \underbrace{f_X(x)}_{\text{derivative of } F_X(x)} \cdot \Delta x \Longrightarrow \underbrace{\frac{\text{prob}}{\text{length}}}_{\text{density}} \cdot \text{length}$

So $f_X(x)$ gives the probability that $X$ falls in a small interval (width $\Delta x$) around $x$.

## Properties of the pdf

1. $f_X(x) \geq 0$
2. $P[a \leq X \leq b] = \int_a^b f_X(x)\,dx$ "area under the curve"
3. $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$
4. $\int_{-\infty}^{\infty} f_X(t)\,dt = 1$ "normalization condition"

**Note:** (by 4) Given any non-negative function (piecewise continuous) $g(x)$ with $\int_{-\infty}^{\infty} g(x)\,dx = c < \infty$ then $f_X(x) = \frac{g(x)}{c}$ is a valid pdf

# BEG CUP, important EE364 pdfs.

B : binomial (and discrete sampling)

E : exponential (gamma) "waiting time"

G : Gaussian / normal "thin-tailed" bell

C : Cauchy "thick-tailed" bell.

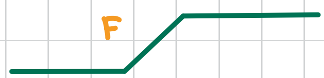
U : Uniform

P : Poisson

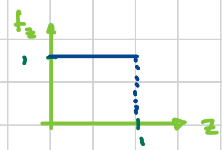
## BEG CUP → Uniform pdf

$X \sim U[a,b]$ : $f_X(x) = \frac{1}{b-a}$ if $x \in [a,b]$, else $= 0$.

$\therefore$ CDF is a "ramp"




Standard Uniform $Z \sim U(0,1)$




Ex: $X \sim U[0,10]$

$$\therefore f_X(x) = \begin{cases} 1/10 & 0 \le x \le 10 \\ 0 & \text{else} \end{cases}$$

$$\underbrace{P[X \le 5]}_{= F_X(5)} = \int_{-\infty}^{5} f_X(x)\,dx$$

$$= \int_0^5 \frac{1}{10}\,dx = \frac{1}{2}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{10}x & 0 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

$$\underbrace{P[2 \le X \le 6]}_{F_X(6) - F_X(2)} = \int_2^6 \frac{1}{10}\,dx = \frac{4}{10} \quad (= 2/5)$$

Later: Beta random variable.

BEG CUP → Gaussian pdf (Normal pdf)

$X \sim N(\mu, \sigma^2)$ : $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

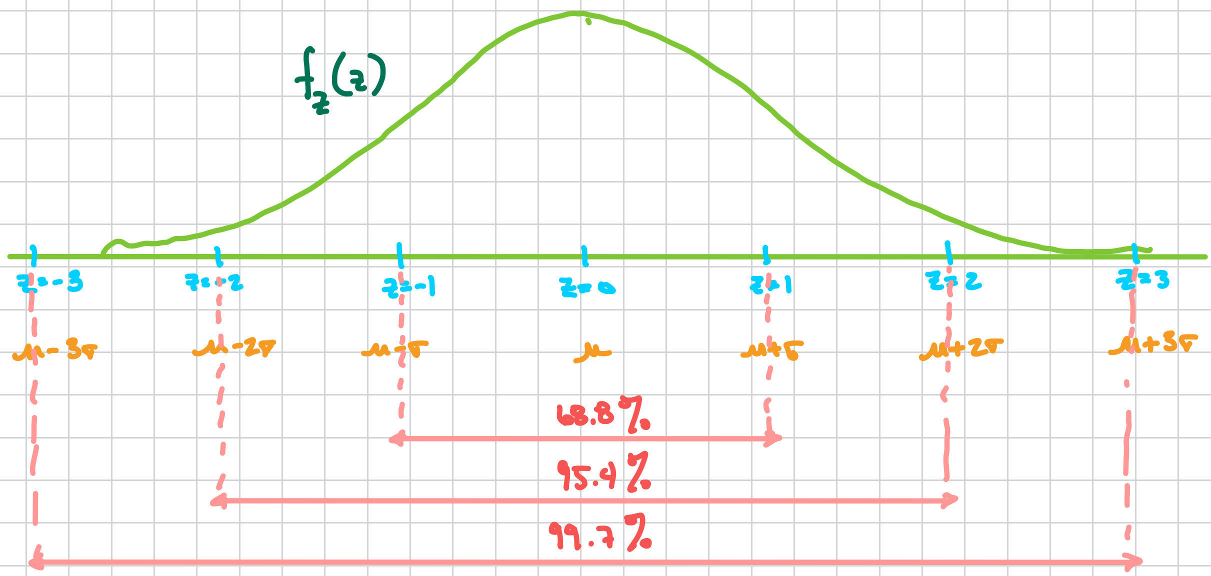
$\mu \in \mathbb{R}$.
$\sigma > 0$
$x \in \mathbb{R}$

mean ($\mu$), variance ($\sigma^2$), $\sigma$: standard deviation

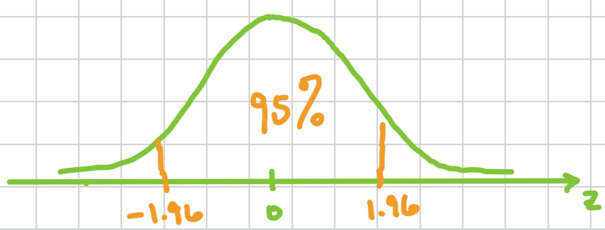
first → Standard Normal:

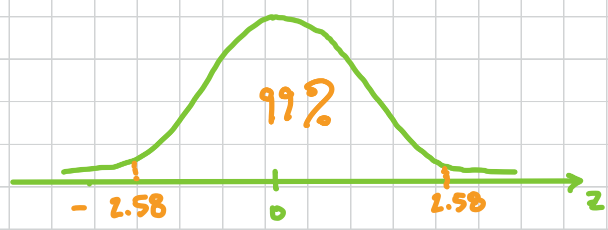
$Z \sim N(0,1)$ : $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$\underbrace{Z \triangleq \frac{X-\mu}{\sigma}}_{\text{standardize}}$




related to sums of random variables




95% ⟷ $z = \pm 1.96$ ($\approx \pm 2\sigma$)

99% ⟷ $z = \pm 2.58$

Fact: Gaussian CDF does not have a closed form (but CDF does exist)

$$F_Z(z) = \phi(z) = \int_{-\infty}^{z} f_Z(t)\,dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt.$$

"Phi-function"

standardize

$$f_X(x) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sigma} f_Z\left(\frac{t-\mu}{\sigma}\right) dt$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} f_Z(z)\,dz$$

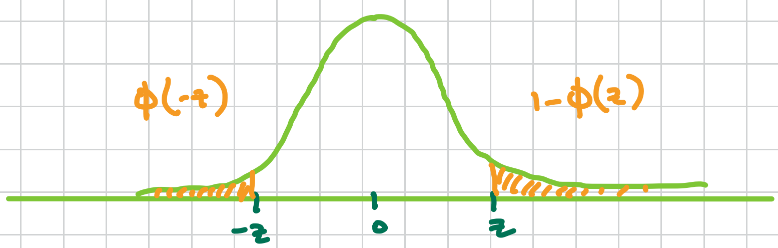
$$= \phi\left(\frac{x-\mu}{\sigma}\right)$$

Ex: Suppose $X \sim N(\mu, \sigma^2)$

$$P[a < X \le b] = \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

Note: $\phi(-z) = 1 - \phi(z)$ (symmetry)

$$P[Z \le -z] = P[Z \ge z]$$




Advanced result: $\lim_{\sigma \to 0} \frac{1}{\sigma} f_Z\left(\frac{x_0-\mu}{\sigma}\right) = \delta(x_0 - \mu)$

# Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Prf: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right)} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy}$

$$\overset{\text{polar}}{=} \sqrt{\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} r\,e^{-r^2}\,dr\,d\theta} \qquad dx\,dy \longmapsto r\cdot dr\,d\theta$$

$$= \sqrt{2\pi}\cdot\sqrt{\int_0^{\infty} r\cdot e^{-r^2}\,dr} = \sqrt{2\pi}\cdot\sqrt{\tfrac{1}{2}} = \sqrt{\pi}$$

QED.

# Normal Table Values

$X \sim N(\mu, \sigma^2)$ CDF: $F_X(x) = \int_{-\infty}^{x} f(w)\,dw$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma}\right)^2} dw$$

no closed form

$Z \sim N(0,1)$ $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2}\,dw$

$\therefore$ use table

Ex: $X \sim N(1,4)$

$$P[X \le 3] = F_X(3) = F_Z\left(\frac{3-1}{2}\right) = \phi(1)$$

$$\underbrace{\frac{X-\mu}{\sigma}}_{Z} \le \frac{3-\mu}{\sigma}$$

$$P[X \le -1] = P\left[\overbrace{\frac{X-1}{2}}^{Z} \le \frac{-1-1}{2}\right] = \phi(-1)$$

$$P[-1 \le X \le 5] = P\left[\frac{-1-1}{2} \le \overbrace{\frac{X-1}{2}}^{Z} \le \frac{5-1}{2}\right] = \phi(2) - \phi(-1)$$

$$P[X \ge 2] = 1 - P[X \le -2] = 1 - P\left[\overbrace{\frac{X-1}{2}}^{Z} \le \frac{-2-1}{2}\right] = \phi(-1.5)$$

Note: Q-function

$$Q(x) = 1 - \phi(x) = P(Z > z)$$

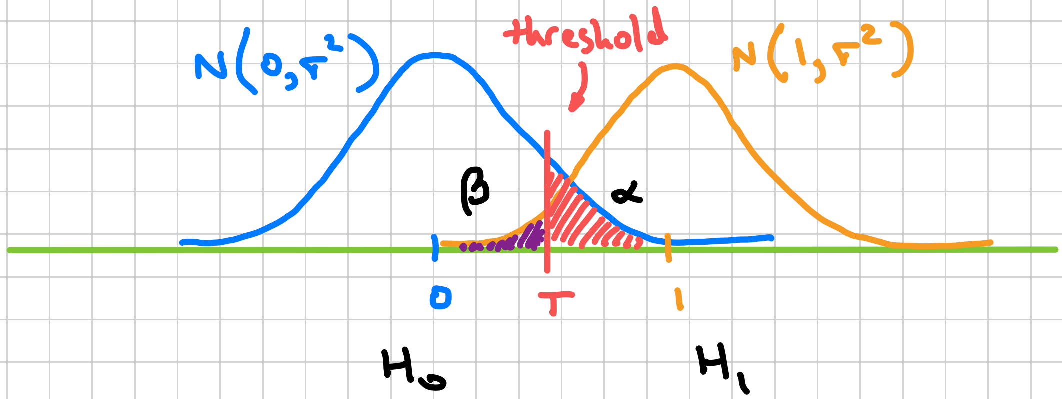
$$= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

for "error function" $\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-w^2}\,dw$

Table 1: Table of the Standard Normal Cumulative Distribution Function $\Phi(z)$

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| -3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| -3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| -3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| -3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| -3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| -2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| -2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| -2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| -2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| -2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| -2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| -2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| -2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| -2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| -2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| -1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| -1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| -1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| -1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| -1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| -1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| -1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| -1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| -1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| -1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| -0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| -0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| -0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| -0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| -0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| -0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| -0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| -0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| -0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| -0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

# Signal Detection




decision rule

$H_0$ if $X \le T$

$H_1$ if $X > T$

$H_0$: no signal $\quad X \sim N(0, \sigma^2)$

$H_a$: signal present $\quad X \sim N(1, \sigma^2)$

$$\alpha \triangleq P[\text{Type 1 error}] = P[\text{reject } H_0 \mid H_0 \text{ true}].$$
$$= P[X > T \mid H_0]$$
$$= P[\text{false alarm}] \qquad \text{false positive}$$

$$\therefore 1 - \alpha = P[\text{correct rejection}].$$

$$\beta \triangleq P[\text{Type 2 error}] = P[\text{accept } H_0 \mid H_a \text{ true}].$$
$$= P[X \le T \mid H_a]$$
$$= P[\text{miss}] \qquad \text{false negative}$$

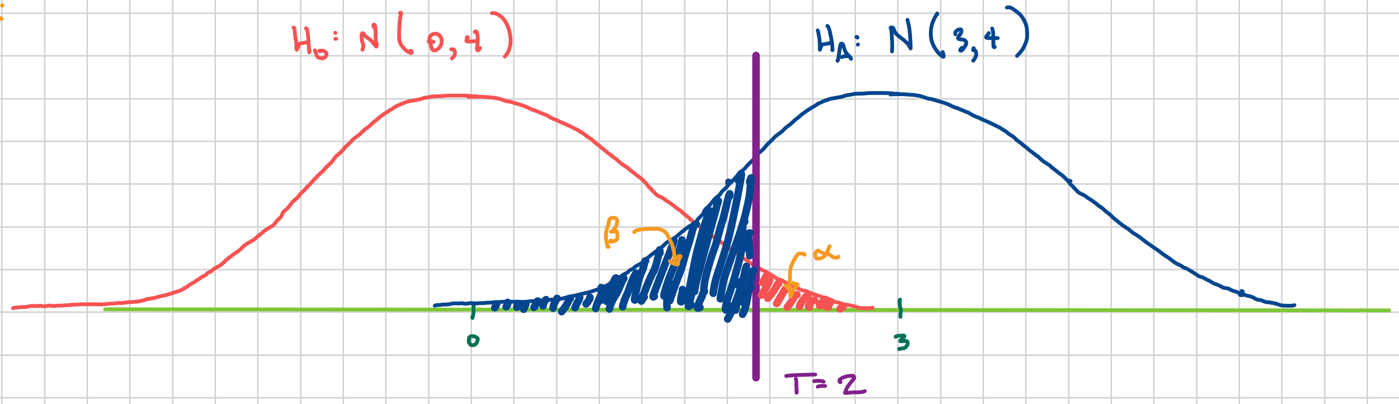
$P[X > T \mid H_a]$

$$\therefore \text{"Power"} \triangleq 1 - \beta = P[\text{Hit}] = P[\text{Correct detection}] = P_D$$

$\therefore \alpha \downarrow \longleftrightarrow \beta \uparrow$

neyman pearson detection:
maximize $P_D$ for fixed false alarm
e.g. $\alpha \le 0.05$.

Ex:




Threshold $T = 2$

$$\alpha = P(\text{type 1 error}) = P\big(\overbrace{\text{reject } H_0}^{X>2} \mid \overbrace{H_0 \text{ true}}^{N(0,4)}\big)$$

$$= P\big(X \geq 2 \mid X \sim N(0,4)\big)$$

$$= P\Big(\underbrace{\frac{X-0}{2}}_{Z\sim N(0,1)} \geq \frac{2-0}{2}\Big)$$

$$= P(Z \geq 1) = 1 - 0.9332 = \boxed{0.06668}$$

$$\beta = P(\text{type 2 error}) = P\big(\overbrace{\text{accept } H_0}^{X<2} \mid \overbrace{H_A \text{ true}}^{N(3,4)}\big)$$

$$= P\Big(\underbrace{\frac{X-3}{2}}_{Z\sim N(0,1)} \leq \frac{2-3}{2}\Big)$$

$$= P\big(Z \leq -\tfrac{1}{2}\big) = \boxed{0.3085}$$

Ex: (at home) repeat, $\underbrace{T=1}_{\substack{\text{move treshold left} \\ 2 \to 1}}$ $\qquad \underbrace{\alpha\uparrow, \ \beta\downarrow}_{\text{tradeoff}}$

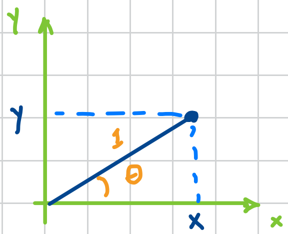
Ex: (at home) find $T$ so that $\alpha = \beta$.

# Trigonometry Review

Pythagoreas : $c^2 = a^2 + b^2$ (right triangle)

Cosine Law : $c^2 = a^2 + b^2 - 2ab\cdot\cos\theta$

(Pythag. $\cos\frac{\pi}{2} = 0$)




$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$$\therefore\ x^2 + y^2 = r^2\cdot\cos^2\theta + r^2\cdot\sin^2\theta = r^2\cdot\left(\sin^2\theta + \cos^2\theta\right)$$

$\therefore r = 1$ ($x^2 + y^2 = 1$) $\longrightarrow$ $\sin^2\theta + \cos^2\theta = 1$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \left(= \frac{y}{x}\right)$$

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

$$\frac{d(\tan\theta)}{d\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = 1 + \tan^2\theta = \sec^2\theta$$

Taylors Series $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\therefore\ e^{i\theta} = \sum_{n=0}^{\infty} \underbrace{\frac{(i\theta)^n}{n!}}_{\frac{i^n\theta^n}{n!}} = \overset{1}{\frac{i^0\theta^0}{0!}} + \overset{i}{\frac{i^1\theta^1}{1!}} + \overset{-1}{\frac{i^2\theta^2}{2!}} + \overset{-i}{\frac{i^3\theta^3}{3!}} + \cdots + \frac{i^n\theta^n}{n!} + \cdots$$

$$= \underbrace{\left(\frac{\theta^0}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\theta^{2n}}{(2n)!} = \cos\theta} + i\underbrace{\left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \cdots\right)}_{\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \sin\theta.}$$

$$= \cos\theta + i\cdot\sin\theta.$$

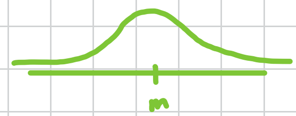
Special case: set $\theta = \pi$ $\quad \therefore \cos\theta = -1$, $\sin\theta = 0$

$$\boxed{\therefore\ e^{i\pi} + 1 = 0}$$

BEG CUP $\longrightarrow$ Cauchy pdf

"Thick tail" bell curve

(infinite variance)




$X \sim C(m, d)$ : $\quad f_X(x) = \dfrac{1}{\pi d \left(1 + \left(\frac{x-m}{d}\right)^2\right)}$

scale or "dispersion" $d > 0$ $\qquad$ location: $m$

CDF: $\quad F = \tan^{-1} + \frac{1}{2}$

$$F_X(x) = P(X \le x) = \frac{1}{\pi} \tan^{-1}\left(\frac{x-m}{d}\right) + \frac{1}{2}$$

Standard Cauchy:

$Z \sim C(0,1)$ : $\quad f_Z(z) = \dfrac{1}{\pi(1+z^2)}$

$S\alpha S$ : Symmetric Alpha-Stable pdf.

family of bell-curves: $\quad 0 < \alpha \le 2$

$\alpha \downarrow \longrightarrow$ thicker tails

$\alpha = 2$ : Gaussian

$\alpha = 1$ : Cauchy

(related: Generalized Central Limit Theorem)

**Thrm:** $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

**Pf:** $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

Let $x = \tan\theta$

$dx = d(\tan\theta) = (1+\tan^2\theta)\, d\theta$

$$\therefore \; = \int_{-\pi/2}^{\pi/2} \frac{1+\tan^2\theta}{1+\tan^2\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} 1 \, d\theta = \theta \Big|_{\theta=-\pi/2}^{\theta=\pi/2} = \pi.$$

QED.

$\therefore \; Z \sim \text{Cauchy}(0,1)$ $\qquad f_z(z) = \dfrac{1}{\pi(1+z^2)}$

$$\therefore \int_{-\infty}^{\infty} f_z(z)\, dz = \int_{-\infty}^{\infty} \frac{1}{\pi(1+z^2)}\, dz = 1.$$

**Defn:** Gamma function $\Gamma(\alpha)$ for $\alpha > 0$.

$$\Gamma(\alpha) = \int_0^\infty \underbrace{x^{\alpha-1}}_{\text{power function}} \underbrace{e^{-x}}_{\text{exponential function}} \, dx$$

Integration by parts

**Thrm:** $\underbrace{\int u \, dv = uv}_{\text{"inverse" chain rule.}} - \int v \, du$ $\quad \left( \int u(x) \, dv(x) = u(x) \cdot v(x) - \int v(x) \, du(x) \right)$

Pf: $(uv)' = u'v + uv' \simeq du \, v + u \, dv$

$\therefore u \, dv = (uv)' - v \, du$

$\therefore \int u \, dv = uv - \int v \, du.$

QED.

**Thrm:** $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

Pf: $\Gamma(\alpha+1) = \int_0^\infty x^{\alpha+1-1} e^{-x} \, dx$ $\quad$ Let $u = x^\alpha$ $\quad dv = e^{-x} \, dx$

$du = \alpha x^{\alpha-1} \, dx$ $\quad v = -e^{-x}$

$= -x^\alpha (e^{-x}) \Big|_{x=0}^{x=\infty} - \int_0^\infty (-e^{-x})(\alpha x^{\alpha-1} \, dx)$

$= -(0-0) + \alpha \underbrace{\int_0^\infty x^{\alpha-1} e^{-x} \, dx}_{\Gamma(\alpha)}$

$= \alpha \cdot \Gamma(\alpha)$

QED.

$\therefore \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$ if $\alpha \in \mathbb{R}^+$

☆ $\therefore \Gamma(n+1) = n!$ if $n \in \mathbb{Z}^+$.

**Thrm:** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Pf. $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx$

Let $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2} x^{-1/2}\, dx$

$= \int_0^\infty x^{-\frac{1}{2}} e^{-x}\, dx$

$= \int_{u=0}^{u=\infty} 2 e^{-u^2}\, du$

$= \underbrace{\int_{-\infty}^{+\infty} e^{-u^2}\, du}_{\text{Gaussian integral}}.$

$= \sqrt{\pi}$

BEG CUP  Exponential pdf.

$X \sim \exp(\theta)$: $f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}$ if $x > 0$, $\theta > 0$.

BEG CUP

$\to \gamma$ — $\chi^2$ — $t$, $F$; Erlang

$\to$ Extreme Value pdfs (EV).

- Weibull
- Frechet
- Gumbel

max$(X_1, \dots, X_n)$

E.V. theorem (Fisher-Tippets)

Gamma pdf:

$X \sim \gamma(\alpha, \theta)$  $x > 0$, $\alpha > 0$, $\theta > 0$

$$f_X(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \cdot \theta^{\alpha}}$$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx.$$

$\therefore$ 3-special cases

1. $\alpha = 1 \longrightarrow$ Exponential ("waiting time") — next, $k=1$
2. $\alpha \in \mathbb{Z}^+ \longrightarrow$ Erlang — $k$ waiting time
3. $\alpha = \frac{r}{2}$, $\theta = 2 \longrightarrow X \sim \chi^2(r)$ for "r degrees of freedom"
   $r \in \mathbb{Z}^+$

Recall: $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

put $y = \frac{x}{\theta}$ $\quad (\theta > 0)$ $\quad \therefore\ dy = \frac{1}{\theta} \cdot dx$

$$\Gamma(\alpha) = \int_{x=0}^{x=\infty} e^{-x/\theta} \cdot \left(\frac{x}{\theta}\right)^{\alpha-1} \cdot \frac{1}{\theta}\, dx$$

$$= \frac{1}{\theta^\alpha} \cdot \int_0^\infty x^{\alpha-1} e^{-x/\theta}\, dx.$$

$\therefore\ X \sim \gamma(\alpha, \theta)$ $\qquad f_x(x) = \dfrac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\, \theta^\alpha}$, $\alpha > 0$, $\theta > 0$, $x \geq 0$

$$\therefore \int_0^\infty \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha) \cdot \theta^\alpha}\, dx = \frac{1}{\Gamma(\alpha)} \cdot \underbrace{\int_0^\infty \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha}\, dx}_{\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

$\therefore\ f_x(x)$ is a valid pdf.

## Chi-Square pdf

$X \sim \gamma(\frac{r}{2}, 2) = \chi^2(r)$ with "r-degrees of freedom"

$$f(x) \triangleq \frac{x^{\frac{r}{2}-1} e^{-x/2}}{\Gamma(r/2)\, 2^{r/2}} \quad \text{if } x > 0 \quad (r > 0)$$

$\therefore X \sim \gamma(\alpha, \theta)$ if $\alpha = \frac{r}{2}$ & $\theta = 2$.

Sums (3) $\sum_{k=1}^{n} X_k \sim \chi^2(\sum_{k=1}^{n} r_k)$ if independent $X_k \sim \chi^2(r_k)$

---

r-even: $\Gamma(\frac{r}{2}) = (\frac{r}{2} - 1)!$ r-odd: $\Gamma(r + \frac{1}{2}) = \frac{(2r)!}{r!\, 4^r}\sqrt{\pi}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$ ← Gaussian integral

$\Gamma(1) = 1$ mass of Exp(1) pdf

$\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ from $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

$\Gamma(2) = 1$

$\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}$

$\Gamma(3) = 2$

$\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$

$\Gamma(4) = 3! = 6$

**Ex:** Find $P[3.25 \le X \le 20.5]$ if $X \sim \chi^2(10)$

Using R: $P[3.25 \le X \le 20.5] = P[X \le 20.5] - P[X \le 3.25]$

$= \text{pchisq}(20.5, 10) - \text{pchisq}(3.25, 10)$

$= 0.9751371 - 0.0250865$

$= 0.9500506$

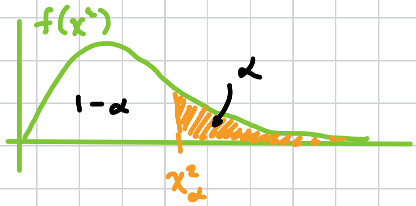
Using table:

$\chi^2_\alpha(10)$ closest to 3.25

$P[X \le 3.25] \approx 1-\alpha = 1-0.975 = 0.025$

$P[X \le 20.5] \approx 1-\alpha = 1-0.025 = 0.975$

$\chi^2_\alpha(10)$ closest to 20.5




$\therefore P[3.25 \le X \le 20.5] = P[X \le 20.5] - P[X \le 3.25]$

$= 0.975 - 0.025$

$= 0.95.$

$r = 10$

$P(X \le 20.5) \approx 1 - \alpha$

$= 1 - 0.025$

$= 0.975$

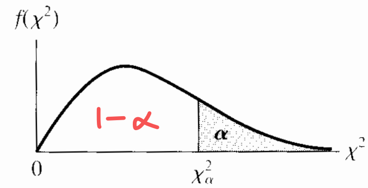
Exact:

$P(X \le 20.5) = 0.9751371$

| Degrees of Freedom | $\chi^2_{.100}$ | $\chi^2_{.050}$ | $\chi^2_{.025}$ | $\chi^2_{.010}$ | $\chi^2_{.005}$ |
|---|---|---|---|---|---|
| 1 | 2.70554 | 3.84146 | 5.02389 | 6.63490 | 7.87944 |
| 2 | 4.60517 | 5.99147 | 7.37776 | 9.21034 | 10.5966 |
| 3 | 6.25139 | 7.81473 | 9.34840 | 11.3449 | 12.8381 |
| 4 | 7.77944 | 9.48773 | 11.1433 | 13.2767 | 14.8602 |
| 5 | 9.23635 | 11.0705 | 12.8325 | 15.0863 | 16.7496 |
| 6 | 10.6446 | 12.5916 | 14.4494 | 16.8119 | 18.5476 |
| 7 | 12.0170 | 14.0671 | 16.0128 | 18.4753 | 20.2777 |
| 8 | 13.3616 | 15.5073 | 17.5346 | 20.0902 | 21.9550 |
| 9 | 14.6837 | 16.9190 | 19.0228 | 21.6660 | 23.5893 |
| 10 | 15.9871 | 18.3070 | 20.4831 | 23.2093 | 25.1882 |
| 11 | 17.2750 | 19.6751 | 21.9200 | 24.7250 | 26.7569 |
| 12 | 18.5494 | 21.0261 | 23.3367 | 26.2170 | 28.2995 |
| 13 | 19.8119 | 22.3621 | 24.7356 | 27.6883 | 29.8194 |
| 14 | 21.0642 | 23.6848 | 26.1190 | 29.1413 | 31.3193 |
| 15 | 22.3072 | 24.9958 | 27.4884 | 30.5779 | 32.8013 |
| 16 | 23.5418 | 26.2962 | 28.8454 | 31.9999 | 34.2672 |
| 17 | 24.7690 | 27.5871 | 30.1910 | 33.4087 | 35.7185 |
| 18 | 25.9894 | 28.8693 | 31.5264 | 34.8053 | 37.1564 |
| 19 | 27.2036 | 30.1435 | 32.8523 | 36.1908 | 38.5822 |
| 20 | 28.4120 | 31.4104 | 34.1696 | 37.5662 | 39.9968 |
| 21 | 29.6151 | 32.6705 | 35.4789 | 38.9321 | 41.4010 |
| 22 | 30.8133 | 33.9244 | 36.7807 | 40.2894 | 42.7956 |
| 23 | 32.0069 | 35.1725 | 38.0757 | 41.6384 | 44.1813 |
| 24 | 33.1963 | 36.4151 | 39.3641 | 42.9798 | 45.5585 |
| 25 | 34.3816 | 37.6525 | 40.6465 | 44.3141 | 46.9278 |
| 26 | 35.5631 | 38.8852 | 41.9232 | 45.6417 | 48.2899 |
| 27 | 36.7412 | 40.1133 | 43.1944 | 46.9630 | 49.6449 |
| 28 | 37.9159 | 41.3372 | 44.4607 | 48.2782 | 50.9933 |
| 29 | 39.0875 | 42.5569 | 45.7222 | 49.5879 | 52.3356 |
| 30 | 40.2560 | 43.7729 | 46.9792 | 50.8922 | 53.6720 |
| 40 | 51.8050 | 55.7585 | 59.3417 | 63.6907 | 66.7659 |
| 50 | 63.1671 | 67.5048 | 71.4202 | 76.1539 | 79.4900 |
| 60 | 74.3970 | 79.0819 | 83.2976 | 88.3794 | 91.9517 |
| 70 | 85.5271 | 90.5312 | 95.0231 | 100.425 | 104.215 |
| 80 | 96.5782 | 101.879 | 106.629 | 112.329 | 116.321 |
| 90 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| 100 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |

*Source:* From Thompson, C. M. "Tables of the percentage points of the $\chi^2$-distribution." *Biometrika*, 1941, Vol. 32, pp. 188–189. Reproduced by permisson of the *Biometrika* Trustees.

**TABLE 8 Critical Values of $\chi^2$**




| Degrees of Freedom | $\chi^2_{.995}$ | $\chi^2_{.990}$ | $\chi^2_{.975}$ | $\chi^2_{.950}$ | $\chi^2_{.900}$ |
|---|---|---|---|---|---|
| 1 | .0000393 | .0001571 | .0009821 | .0039321 | .0157908 |
| 2 | .0100251 | .0201007 | .0506356 | .102587 | .210720 |
| 3 | .0717212 | .114832 | .215795 | .351846 | .584375 |
| 4 | .206990 | .297110 | .484419 | .710721 | 1.063623 |
| 5 | .411740 | .554300 | .831211 | 1.145476 | 1.61031 |
| 6 | .675727 | 0.872085 | 1.237347 | 1.63539 | 2.20413 |
| 7 | .989265 | 1.239043 | 1.68987 | 2.16735 | 2.83311 |
| 8 | 1.344419 | 1.646482 | 2.17973 | 2.73264 | 3.48954 |
| 9 | 1.734926 | 2.087912 | 2.70039 | 3.32511 | 4.16816 |
| 10 | 2.15585 | 2.55821 | 3.24697 | 3.94030 | 4.86518 |
| 11 | 2.60321 | 3.05347 | 3.81575 | 4.57481 | 5.57779 |
| 12 | 3.07382 | 3.57056 | 4.40379 | 5.22603 | 6.30380 |
| 13 | 3.56503 | 4.10691 | 5.00874 | 5.89186 | 7.04150 |
| 14 | 4.07468 | 4.66043 | 5.62872 | 6.57063 | 7.78953 |
| 15 | 4.60094 | 5.22935 | 6.26214 | 7.26094 | 8.54675 |
| 16 | 5.14224 | 5.81221 | 6.90766 | 7.96164 | 9.31223 |
| 17 | 5.69724 | 6.40776 | 7.56418 | 8.67176 | 10.0852 |
| 18 | 6.26481 | 7.01491 | 8.23075 | 9.39046 | 10.8649 |
| 19 | 6.84398 | 7.63273 | 8.90655 | 10.1170 | 11.6509 |
| 20 | 7.43386 | 8.26040 | 9.59083 | 10.8508 | 12.4426 |
| 21 | 8.03366 | 8.89720 | 10.28293 | 11.5913 | 13.2396 |
| 22 | 8.64272 | 9.54249 | 10.9823 | 12.3380 | 14.0415 |
| 23 | 9.26042 | 10.19567 | 11.6885 | 13.0905 | 14.8479 |
| 24 | 9.88623 | 10.8564 | 12.4011 | 13.8484 | 15.6587 |
| 25 | 10.5197 | 11.5240 | 13.1197 | 14.6114 | 16.4734 |
| 26 | 11.1603 | 12.1981 | 13.8439 | 15.3791 | 17.2919 |
| 27 | 11.8076 | 12.8786 | 14.5733 | 16.1513 | 18.1138 |
| 28 | 12.4613 | 13.5648 | 15.3079 | 16.9279 | 18.9392 |
| 29 | 13.1211 | 14.2565 | 16.0471 | 17.7083 | 19.7677 |
| 30 | 13.7867 | 14.9535 | 16.7908 | 18.4926 | 20.5992 |
| 40 | 20.7065 | 22.1643 | 24.4331 | 26.5093 | 29.0505 |
| 50 | 27.9907 | 29.7067 | 32.3574 | 34.7642 | 37.6886 |
| 60 | 35.5346 | 37.4848 | 40.4817 | 43.1879 | 46.4589 |
| 70 | 43.2752 | 45.4418 | 48.7576 | 51.7393 | 55.3290 |
| 80 | 51.1720 | 53.5400 | 57.1532 | 60.3915 | 64.2778 |
| 90 | 59.1963 | 61.7541 | 65.6466 | 69.1260 | 73.2912 |
| 100 | 67.3276 | 70.0648 | 74.2219 | 77.9295 | 82.3581 |

$r = 10$

$P(X \le 3.25) = 1 - \alpha$
$= 1 - 0.975$
$= 0.025$

Exact:
$P(X \le 3.25) = 0.250865$

$X \sim \chi^2(r)$ — $r$. d.f.

$P[X \le x] \approx 1 - \alpha$ for $\chi^2_\alpha$ closest to $x$.

Beta random variable

$X \sim \text{Beta}(\alpha, \beta): \ f_X(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

if $0 < x < 1$ (else $= 0$)

$\therefore \boxed{\alpha = \beta = 1 \longrightarrow X \sim U[0,1].}$

(n-dim, Dirichlet pdf)

Defn: Beta function $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ $\quad \alpha > 0$, $\beta > 0$

$= \dfrac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ (see below)

Thrm: $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du \qquad \alpha>0,\ \beta>0$

$= \dfrac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Prf: $\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x} x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y} y^{\beta-1}\,dy$

$\overset{\text{Fubini}}{=} \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)} x^{\alpha-1} y^{\beta-1}\,dx\right] dy$

$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f\big(x(u,v),\, y(u,v)\big)\,dx\,dy$

Def: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$ double substitution

Given: $0<x<\infty$ and $0<y<\infty$

$\therefore\ 0 < x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u>0$

$\therefore\ x>0 \longrightarrow uv>0$

$\therefore\ v>0$ since $u>0$

$\therefore\ u<\infty$ since $x = u\cdot v < \infty$ and $v>0$

$\therefore\ 0<u<\infty \longleftarrow$ first limit of integration

$\therefore\ y>0 \longrightarrow u(1-v)>0 \qquad \therefore\ 1-v>0$ (since $u>0$)

$\therefore\ v<1 \qquad \therefore\ 0<v<1 \longleftarrow$ second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\; u \cdot du\, dv.$$

since $\underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute determinant of Jacobian matrix}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$ since $x = u\cdot v$, $y = u - uv$

$$= \left| -\cancel{vu} - u + \cancel{vu} \right| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}(uv)^{\alpha-1}\big(u(1-v)\big)^{\beta-1} u \; du\, dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1} v^{\alpha-1} u^{\beta-1}(1-v)^{\beta-1} u^{1} \, du\, dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1+\beta-1+1} du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1} dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{(\alpha+\beta)-1} du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1} dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED